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G-COVERING SUBGROUP SYSTEMS FOR THE CLASSES OF SUPERSOLUBLE AND NILPOTENT GROUPS

BY

WENBIN GUO*

Department of Mathematics, Xuzhou Normal University
221009 Xuzhou, P.R. China
and
Department of Mathematics, University of Science and Technology of China
230026 Hefei, P. R. China
e-mail: yzgwb@pub.yz.jsinfo.net

AND

K. P. SHUM†

Faculty of Science, The Chinese University of Hong Kong
Shatin, Hong Kong, P.R. China (SAR)
e-mail: kpshum@math.cuhk.edu.hk

AND

ALEXANDER SKIBA

Department of Mathematics, Fr.Skaryna Gomel State University
246019 Gomel, Belarus
e-mail: skiba@gsu.unibel.by

ABSTRACT

Let $\mathcal{F}$ be a class of groups and G a group. We call a set Σ of subgroups of G a G-covering subgroup system for the class $\mathcal{F}$ (or directly a $\mathcal{F}$-covering subgroup system of G) if $G \in \mathcal{F}$ whenever every subgroup in Σ is in $\mathcal{F}$. In this paper, we provide some nontrivial sets of subgroups of a finite group G which are simultaneously G-covering subgroup systems for the classes of supersoluble and nilpotent groups.

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1. Introduction

Let $\mathcal{F}$ be a class of groups and G a group. Then we call a set Σ of subgroups of G a G-covering subgroup system for the class $\mathcal{F}$ (or directly a $\mathcal{F}$-covering subgroup system of G) if $G \in \mathcal{F}$ whenever every subgroup in Σ is in $\mathcal{F}$. Clearly, the set of all finitely generated subgroups of a group G is a G-covering subgroup system for the class of all abelian groups. Also, by the well known Mal'cev local theorem [10] (see also [9; Section 22]), we know that the set of all finitely generated subgroups of a group G is a G-covering subgroup system for many other important classes of groups. However, in the theory of finite groups, we know even less examples of such kind and most of these examples are confined in the classes of nilpotent and p-nilpotent groups. For example, if P is a Sylow p-subgroup of a finite group G, where p is odd, then by the famous Thompson J-theorem, the set $\{N_G(J(P)), C_G(Z(P))\}$ is a G-covering subgroup system for the class of all p-nilpotent groups. According to [1], the set of all normalizers of all Sylow subgroups of a finite group G is a G-covering subgroup system for the class $\mathcal{N}$ of all nilpotent groups and, by Fedri and Serens [5], there are groups G in which the set of all normalizers of all Sylow subgroups is not a G-covering subgroup system for the class $\mathcal{U}$ of all supersoluble groups. Another example is the set Σ of all biprimary subgroups of a group G. This set forms a $\mathcal{N}$-covering subgroup system of G. Indeed, it is well known that every finite minimal nonnilpotent group (that is, a nonnilpotent group in which every proper subgroup is nilpotent) is biprimary. Hence, a group G is nilpotent if every subgroup in Σ is nilpotent. We note that this system is in general not a $\mathcal{U}$-covering subgroup system of G, because there are finite nonsupersoluble groups in which all biprimary subgroups are supersoluble. On the other hand, from the well known Huppert's results on minimal nonsupersoluble groups [7], we see that the set of all subgroups H of a finite group G with $|\pi(H)| \leq 3$ forms a $\mathcal{U}$-covering subgroup system of G, where $\pi(H)$ is the set of all prime divisors of the order $|H|$ of H.

In this paper, we will give some new examples of $\mathcal{U}$-covering subgroup systems for finite groups. One of our main results is (see Theorem 3.7): *Let G be a group. Assume that there exists a set $\mathcal{F}$ of subgroups of G having the following property: For every maximal subgroup M of any Sylow subgroup of G, either M is normal in G or $\mathcal{F}$ contains a supplement of M in G. Then $\mathcal{F}$ forms a G-covering subgroup system for the class of supersoluble groups.* In addition, we also consider other sets of subgroups of a finite group G which are simultaneously G-covering subgroup systems for both classes of supersoluble and nilpotent groups. In fact, we give some new conditions for a group G to be supersoluble or nilpotent, and also we provide a new approach to study the properties of groups.

The reader is referred to Doerk and Hawkes [4] for terminology and definitions, if necessary.

2. Preliminaries

All groups considered in this paper are finite. For the sake of convenience, we first cite some known results which will be frequently used in the proofs. These results may be found, for example, in [4; Chapter A and Chapter 1].

LEMMA 2.1 (Wielandt): *If a group G has three soluble subgroups T_1, T_2, T_3 whose indices $|G:T_1|$, $|G:T_2|$, $|G:T_3|$ are pairwise coprime, then G is soluble.*

LEMMA 2.2: *Let N be a normal subgroup of a group G such that $N/N\cap\Phi(G)$ is nilpotent. Then N is also nilpotent.*

LEMMA 2.3 (Huppert): *A group G is supersoluble if and only if every maximal subgroup of G has prime index in G.*

LEMMA 2.4: *Let H/K be a chief factor of a group G. Then $|H/K|$ is a prime p if and only if $G/C_G(H/K)$ is a cyclic group whose order divides $p-1$.*

We use the symbol $Soc(G)$ to denote the product of all minimal normal subgroups of G. We recall that the product of all nilpotent normal subgroups of a group G is the Fitting subgroup of G and is denoted by $F(G)$. Usually, we use the internal structure of $F(G)$ and its location among others subgroups of a soluble group G to determine the properties of the base group G. However, if the group G is not soluble, then the Fitting subgroup of G may be trivial, and in this case, various generalizations of $F(G)$ are possible. In particular, we consider the subgroup $\overline{F}(G)$ of G which is a subgroup of G satisfying the conditions $\Phi(G)\subseteq\overline{F}(G)$ and $\overline{F}(G)/\Phi(G)=Soc(G/\Phi(G))$. We cite the following results which will be useful in the sequel.

LEMMA 2.5 (Gaschütz): *Let G be a soluble group. Then*

$$F(G)/\Phi(G)=Soc(G/\Phi(G)).$$

LEMMA 2.6: *Let H/K be a chief factor of a group G. Then $F(G)\subseteq C_G(H/K)$.*

LEMMA 2.7: *Let G be a soluble group. Then $C_G(F(G))\subseteq F(G)$.*

LEMMA 2.8: *Let $N \leq K \leq Soc(G)$, where $N, K \trianglelefteq G$. Then there is a normal subgroup T of G such that $K = N \times T$.*

A group G is called primitive if G contains a maximal subgroup M such that $M_G = 1$, where M_G is the core of M in G. For primitive groups, we have the following results:

LEMMA 2.9 (Baer): *Let G be a primitive group. If G contains an abelian minimal normal subgroup R, then $R = Soc(G)$.*

LEMMA 2.10: *Let H/K be a chief factor of a group G. If H/K is a p-group, then $O_p(G/C_G(H/K)) = 1$.*

LEMMA 2.11: *If G is a supersoluble group, p and q are maximal and minimal prime divisors of $|G|$ respectively, then a Sylow p-subgroup of G is normal in G and G is q-nilpotent.*

LEMMA 2.12 (Cunihin [2]): *If $G = AB$ with $D \trianglelefteq A$ and $D \subseteq B$, then $D \subseteq B_G$.*

LEMMA 2.13 (Hall): *Let G be a group. Then G is soluble if and only if every Sylow subgroup of G is complemented in G.*

LEMMA 2.14: *If $1 \neq N \trianglelefteq G$ and $\Phi(G) = 1$, then $Soc(N) \subseteq Soc(G)$.*

We use $\pi(G)$ to denote the set of all primes dividing the order $|G|$ of the group G. Also, we call that a subgroup T of a group G a supplement of a subgroup H in G if $G = HT$.

3. Results

LEMMA 3.1: *Let N and L be normal subgroups of a group G. Let P/L be a Sylow p-subgroup of NL/L and M/L a maximal subgroup of P/L. If P_p is a Sylow p-subgroup of $P \cap N$, then P_p is a Sylow p-subgroup of N such that $D = M \cap N \cap P_p$ is a maximal subgroup of P_p and $M = LD$.*

Proof: Since $P \leq NL$ and $L \leq P$, we have $P = P \cap NL = L(P \cap N)$. This implies that $P/L = L(P \cap N)/L \simeq (P \cap N)/(L \cap N \cap P) = (P \cap N)/(L \cap N) \leq N/(L \cap N)$. Since $|NL/L| = |N/L \cap N|$, $|P/L| = |(P \cap N)/(L \cap N)|$ and since P/L is a Sylow p-subgroup of NL/L, $|N/(L \cap N) : (P \cap N)/(L \cap N)|$ is not divisible by p. This shows that $(P \cap N)/(L \cap N)$ is a Sylow p-subgroup of $N/(L \cap N)$. Analogously, since $M/L = (M \cap N)L/L \simeq (M \cap N)/(L \cap N)$, we have

$$|NL/L : M/L| = |N/(L \cap N) : (M \cap N)/(L \cap N)|.$$

But $(M \cap N)/(L \cap N) \leq (P \cap N)/(L \cap N)$, and hence $(M \cap N)/(L \cap N)$ is a maximal subgroup of $(P \cap N)/(L \cap N)$. Let P_p be a Sylow p-subgroup of $P \cap N$. Then $P_p(L \cap N)/(L \cap N) = (P \cap N)/(L \cap N)$. Since p does not divide $|N/(L \cap N) : (P \cap N)/((L \cap N)|$, we see that P_p is a Sylow p-subgroup of the group N. Since $M \cap N = M \cap N \cap P_p(L \cap N) = (L \cap N)(M \cap N \cap P_p)$, we deduce that

$$\begin{aligned} p =& |(P \cap N)/(L \cap N) : (M \cap N)/(L \cap N)| \\ =& |(P \cap N) : (M \cap N)| \\ =& |P_p(L \cap N) : (L \cap N)(M \cap N \cap P_p)| \\ =& \frac{|P_p(L \cap N)|}{|(L \cap N)(M \cap N \cap P_p)|} \\ =& \frac{|P_p||L \cap N||L \cap N \cap M \cap N \cap P_p|}{|L \cap N||M \cap N \cap P_p||P_p \cap L \cap N|} \\ =& \frac{|P_p|}{|M \cap N \cap P_p|}. \end{aligned}$$

Hence $D = M \cap N \cap P_p$ is a maximal subgroup of P_p. From the above, we see that $M \cap N = (L \cap N)D$, and so $M = M \cap LN = L(M \cap N) = L(L \cap N)D = LD$. ∎

THEOREM 3.2: *Let G be a soluble group having a normal subgroup N such that G/N is supersoluble. Assume that there exists a set $\mathcal{F}$ of subgroups of G having the following property: For every maximal subgroup M of any Sylow subgroup of $F(N)$, either M is normal in G or $\mathcal{F}$ contains a supplement of M in G. Then $\mathcal{F}$ forms a G-covering subgroup system for the class of supersoluble groups.*

Proof: Assume that the theorem is false and we let G be a counterexample with minimal order.

Let $\Phi = \Phi(G)$. Then we consider the quotient group G/Φ. We first show that the hypotheses of the theorem are true for G/Φ. Let $A/\Phi = F(N\Phi/\Phi)$. Then $A = A \cap N\Phi = \Phi(A \cap N)$. Since A/Φ is a nilpotent normal subgroup of G/Φ, A is a nilpotent normal subgroup of G, by Lemma 2.2. Hence, we have $A \cap N \leq F(N)$. On the other hand, since $F(N)/F(N) \cap \Phi \simeq F(N)\Phi/\Phi \leq F(N\Phi/\Phi)$, we have $F(N) \subseteq A$. Consequently, we obtain $A \cap N = F(N)$, and thereby $F(N\Phi/\Phi) = A/\Phi = (A \cap N)\Phi/\Phi = F(N)\Phi/\Phi$.

Now let P/Φ be a Sylow p-subgroup of A/Φ and M/Φ a maximal subgroup of P/Φ and P_p a Sylow p-subgroup of $P \cap F(N)$. Then by Lemma 3.1, P_p is a Sylow p-subgroup of $F(N)$ and $D = M \cap F(N) \cap P_p$ is a maximal subgroup of P_p. By the choice of G, either G has a supersoluble subgroup T such that $DT = G$ or D is normal in G. By Lemma 3.1, we have $M = D\Phi$. Hence in the latter case, $D\Phi$ is also normal in G, and so $M/\Phi = D\Phi/\Phi$ is normal in G/Φ. Now, we consider the former case $DT = G$. In this case, we see that $T\Phi/\Phi \simeq T/T \cap \Phi$, which is a supersoluble subgroup of G/Φ such that $(T\Phi/\Phi)(M/\Phi) = G/\Phi$. This shows that the group G/Φ contains a normal subgroup $N\Phi/\Phi$ such that each maximal subgroup of every Sylow subgroup of $F(N\Phi/\Phi) = F(N)\Phi/\Phi$ has either a supersoluble supplement in G/Φ or it is normal in G/Φ. Clearly, $(G/\Phi)/(N\Phi/\Phi) \simeq G/N\Phi \simeq (G/N)/(N\Phi/N)$ is a supersoluble group.

If $\Phi \neq 1$, then $|G/\Phi| < |G|$ and, by the choice of G, we see that G/Φ is supersoluble and hence G is supersoluble by Lemma 2.2. However, by our assumption on G, this is absurd. Consequently, we have $\Phi(G) = 1$, and by Lemma 2.5, $F(G) = R_1 \times R_2 \times \cdots \times R_t$, where $R_1, R_2, \ldots, R_t$ are minimal normal subgroups of G. Assume that, for all $i \in \{1, 2, \ldots, t\}$, the order of the minimal normal subgroup R_i of G is a prime number. Then, by Lemma 2.4, $G/C_G(R_i)$ is abelian. Consider $C = \bigcap_{i=1}^{t} C_G(R_i)$. Then, it is clear that $C = C_G(F(G))$. Clearly, $F(G) \leq C$. But by Lemma 2.7, $C \leq F(G)$. Hence $C = F(G)$, and so $G/F(G)$ is an abelian group. This shows that every maximal subgroup E of G containing $F(G)$ is normal in G, and hence $|G : E|$ is a prime. On the other hand, if E is a maximal subgroup of G such that $F(G) \not\subseteq E$, then $G = R_iE$ for some $i \in \{1, 2, \ldots, t\}$, and so $|G : E| = |R_i|$ is a prime number, because every R_i is a group of prime order. Thus, the group G is supersoluble by Lemma 2.3. This contradiction shows that there exists an index i such that $|R_i| \neq p$ for all primes p.

Without loss of generality, we may assume that $i = 1$. In this case, we let $R_1 \not\leq N$. Then $R_1 \cap N = 1$, and so R_1 is isomorphic to the chief factor R_1N/N of G/N. But, by our hypotheses, G/N is a supersoluble group, and consequently, $|R_1| = |R_1N/N|$ is a group of prime order. This contradiction shows that $R_1 \subseteq N$, and so $R_1 \subseteq F(N)$. Let R_1 be a p-group and P a Sylow p-subgroup of $F(N)$. Because P is a characteristic subgroup of $F(N)$ and $F(N)$ is a characteristic subgroup of $N \trianglelefteq G$, we see that P is a normal subgroup of G and, by Lemma 2.8, P can be expressed as a direct product of some minimal normal subgroups of G. Without loss of generality, we can assume that $P = R_1 \times R_2 \times \cdots \times R_n$.

Let M be a maximal subgroup of R_1. Clearly, $n > 1$. Since $|R_1 : M| = p$, we have $|P : MR_2 \cdots R_n| = p$, and so we see that $MR_2 \cdots R_n$ is a maximal subgroup of P. By the choice of G, we see that $MR_2 \cdots R_n$ is either normal in G or there is a supersoluble subgroup T of G such that $T(MR_2 \cdots R_n) = G$. Suppose $MR_2 \cdots R_n \trianglelefteq G$. Then, we have $(MR_2 \cdots R_n) \cap R_1 = M(R_1 \cap R_2 \cdots R_n) = M \trianglelefteq G$. Now, R_1 is a minimal normal subgroup of G with $|R_1| \neq p$, a contradiction. Hence, $MR_2 \cdots R_n$ must have a supersoluble supplement T in G. Assume that $R_1 \leq T$. Then $TMR_2 \cdots R_n = TR_2 \cdots R_n$, and thereby $G/R_2 \cdots R_n = T(R_2 \cdots R_n)/(R_2 \cdots R_n) \simeq T/T \cap (R_2 \cdots R_n)$. This leads to $G/R_2 \cdots R_n$, a supersoluble group. In this case, we see that $R_1 \simeq P/R_2 \cdots R_n$ is a group of prime order, a contradiction. Thus $R_1 \not\leq T$. On the other hand, if $R_2 \cdots R_n \leq T$, then we have $G = TMR_2 \cdots R_n = TM$, and hence $|G : T| \leq |M| < |R_1|$. But $R_1 T = G$, and so we have $|G : T| = |R_1|$, a contradiction.

Let E be a minimal normal subgroup of G contained in $R_2 \cdots R_t$. Then, since

$$\bigcap_{i=2}^{n} R_2 \cdots R_{i-1} R_{i+1} \cdots R_n = 1,$$

there exists an index j such that

$$R_2 \cdots R_{j-1} R_j R_{j+1} \cdots R_n = R_2 \cdots R_{j-1} E R_{j+1} \cdots R_n.$$

Thus we may suppose without loss of generality that there is an index $2 \leq i < n$ such that for every minimal normal subgroup L of G contained in $R_i \cdots R_n$ we have $L \not\leq T$, and that $R_k \leq T$ for all $1 < k < i$. Now let $D = (R_1 R_i \cdots R_n) \cap T$. We claim that $D \neq 1$. If $D = 1$, then by $G = TR_1 \cdots R_n = TR_1 R_i \cdots R_n$, we have $|G : T| = |R_1||R_i| \cdots |R_n|$. On the other hand, because $G = TMR_2 \cdots R_n = TMR_i \cdots R_n$ we have $|G : T| \leq |M||R_i| \cdots |R_n|$, a contradiction. Consequently, $D \neq 1$ and our claim holds. Now, let L be a minimal normal subgroup of G contained in D. Since $L \leq T$, we have $L \not\leq R_i \cdots R_n$. But $L \leq R_1 R_i \cdots R_n$, and hence we have $LR_i \cdots R_n = R_1 R_i \cdots R_n$. This leads to $G = TR_1 R_2 \cdots R_n = R_2 \cdots R_{i-1} T R_1 R_i \cdots R_n = TLR_i \cdots R_n = TR_i \cdots R_n$, and hence $G/R_i \cdots R_n \simeq T/(T \cap R_i \cdots R_n)$. This shows that $G/R_i \cdots R_n$ is a supersoluble group. Thus, $R_1 \simeq R_1 R_i \cdots R_n / R_i \cdots R_n$ is a group of prime order. This contradiction completes the proof of Theorem 3.2. ∎

COROLLARY 3.3: *Let G be a group having a normal subgroup N such that G/N is supersoluble. Assume that there exists a set $\mathcal{F}$ of subgroups of G having the following property: For every maximal subgroup M of any Sylow subgroup of $\overline{F}(N)$ the set $\mathcal{F}$ contains a supplement of M in G. Then $\mathcal{F}$ forms a G-covering subgroup system for the class of supersoluble groups.*

Proof: In view of Theorem 3.2, we first show that N is soluble. Let $\pi(N) = \{p_1, p_2, \dots, p_t\}$. Since every biprimary group is soluble, we may assume that $t \geq 3$. Let P_i be a Sylow p_i-subgroup of $\overline{F}(N)$ and T_i a supersoluble subgroup of G such that $T_iM_i = G$ for some maximal subgroup M_i of P_i. To prove that N is a soluble group, we first assume that $\overline{F}(N) = F(N)$ and $p_i \in \pi(F(N))$. Since P_i is a characteristic subgroup of $F(N) \trianglelefteq N$, we see that $P_i \trianglelefteq N$. Since $N = N \cap T_iM_i = N \cap P_iT_i = P_i(N \cap T_i)$, we deduce that $N/P_i \simeq (N \cap T_i)/(N \cap T_i \cap P_i)$. Thus, N/P_i is a soluble group, and hence N is a soluble group. Now, we suppose that $\overline{F}(N) \neq F(N)$. Then $\overline{F}(N)$ is not soluble, and so $|\pi(\overline{F}(N))| \geq 3$. Without loss of generality, we may assume that $p_1, p_2, p_3 \in \pi(\overline{F}(N))$. In this case, G has three supersoluble subgroups T_1, T_2, T_3 whose indices $|G : T_1|$, $|G : T_2|$, $|G : T_3|$ are pairwise coprime (here we note that if for some $i \in \{1, 2, 3\}$ the subgroup P_i has a prime order, then $M_i = 1$, and in this case $G = T_iM_i = T_i$ is a supersoluble group). Hence, by Lemma 2.1, G is soluble. ∎

Corollary 3.4: *Let G be a group having a normal subgroup N such that G/N is nilpotent. Assume that there exists a set $\mathcal{F}$ of subgroups of G having the following property: For every maximal subgroup M of any Sylow subgroup of $F(N)$ the set $\mathcal{F}$ contains a supplement of M in G. Then $\mathcal{F}$ forms a G-covering subgroup system for the class of nilpotent groups.*

Proof: Assume that this corollary is false and let G be a counterexample with minimal order. By Theorem 3.2, G is supersoluble and we may assume that $\Phi(G) = 1$ (see the proof of Theorem 3.2). It is also clear that $F(N) = O_p(N) = Soc(N)$ for some prime p. Let $\{R_1, R_2, \dots, R_t\}$ be the set of all minimal normal subgroups of G contained in N. Then, we have $|R_1| = |R_2| = \cdots = |R_t| = p$. Since by Lemma 2.14, $Soc(N) \subseteq Soc(G)$, by Lemma 2.8, we see that for each $i \in \{1, 2, \dots, t\}$ there exists a normal subgroup M_i of G such that $F(N) = M_i \times R_i$. Since M_i is evidently a maximal subgroup in $F(N)$, by the choice of G, there is a nilpotent subgroup T_i of G such that $M_iT_i = G$. This leads to G/M_i is nilpotent. However, since $\bigcap_{i=1}^{t} M_i = 1$, G is a nilpotent group. This contradiction completes the proof. ∎

Corollary 3.5: *Every set of subgroups of a group G containing at least one supplement of T in G, for every maximal subgroup T of any Sylow subgroup of $\overline{F}(G)$, forms a G-covering subgroup system for both classes of nilpotent and supersoluble groups.*

COROLLARY 3.6: *Let G be a soluble group. Let N be a normal subgroup of G with supersoluble quotient. If every maximal subgroup of any Sylow subgroup of $F(N)$ is normal in G, then G is supersoluble.*

THEOREM 3.7: *Let G be a group having a normal subgroup N such that G/N is supersoluble. Assume that there exists a set $\mathcal{F}$ of subgroups of G having the following property: For every maximal subgroup M of any Sylow subgroup of N, either M is normal in G or $\mathcal{F}$ contains a supplement of M in G. Then $\mathcal{F}$ forms a G-covering subgroup system for the class of supersoluble groups.*

Proof: Assume that the theorem is false and let G be a counterexample with minimal order.

Let R be an abelian minimal normal subgroup of G and R_1 a maximal subgroup of R. Assume that there is a supersoluble subgroup T of G such that $TR_1 = G$. Then $|G : T| \leq |R_1| < |R|$. Since R is abelian and $TR = G$, we have $T \cap R \trianglelefteq G$. If $T \cap R = R$, then $G = RT = T$ is a supersoluble group; however, this is not true by the choice of G. Hence $T \cap R \neq R$, and by the minimality of R we have $T \cap R = 1$, i.e., $|G : T| = |R|$. This contradiction shows that for every maximal subgroup R_1 of R and for every supersoluble subgroup T of G, we have $TR_1 \neq G$. Thus, in particular, we have $R \neq N$, and so RN/R is a nontrivial normal subgroup of G/R such that the factor group $(G/R)/(RN/R) \simeq G/RN \simeq (G/N)/(RN/N)$ is supersoluble.

Let P/R be a Sylow q-subgroup of RN/R and M/R a maximal subgroup of P/R. Let P_q be a Sylow q-subgroup of $P \cap N$. Then by Lemma 3.1, P_q is a Sylow q-subgroup of N and $M = RB$, where $B = M \cap N \cap P_q$ is a maximal subgroup of P_q. This leads to B is either normal in G or B has a supersoluble supplement T in G. In the former case, $M/R = BR/R \trianglelefteq G/R$. In the latter case, we see that $(M/R)(TR/R) = MT/R = G/R$ and $TR/R \simeq T/R \cap T$ is supersoluble. Consequently, by the choice of G, we conclude that G/R is a supersoluble group. Since the class of all supersoluble groups is a saturated formation, we see that $R \nleq \Phi(G)$ so that G does not contain an abelian minimal normal subgroup which is different from R.

We now claim that G is soluble. It is clear that $|\pi(G)| > 1$. Let r and q be respectively the minimal and the maximal numbers in the set $\pi(G)$, and let P be a Sylow r-subgroup of G. Assume that $|P| = r$. It is not difficult to show that in this case the group G has a normal complement M with respect to P. By the choice of G, this complement is soluble and G is soluble. Assume then that $|P| > r$. If some maximal subgroup of P is normal in G then we can apply induction and complete the proof of our claim. We now assume that all the maximal subgroups M_i of P are nonnormal in G so that they have the corresponding supersoluble supplements T_i in G.

Let M_j be a maximal subgroup of P with a supersoluble supplement T_j. Then, each coset of M_j in P has a representative from T_j. Now, we can pick a Sylow q-subgroup Q of G which is contained in T_j, and by Lemma 2.11 we have $Q \trianglelefteq T_j$. This shows that each coset of M_j in P has a representative which normalizes Q. Now let M_k be another maximal subgroup of P with a supersoluble supplement T_k. Clearly, this supplement contains a Sylow q-subgroup Q_1 of G, and since P acts transitively on the set $Syl_q(G)$, we can write $Q_1 = Q^{p_1}$, where $p_1 \in P$. Since $Q^{p_1} \lhd T_k$, each coset of M_k in P has a representative which normalizes Q^{p_1}. In other words, M_k has a transversal in P in which each element is of the form g^{p_1}, where g normalizes Q.

We have proved that each maximal subgroup M_i of P has a transversal S_i in P in which each element is of the form g^{p_1}, where g normalizes Q and $p_1 \in P$. Now, let S be the union of all these transversals S_i. Then, we can easily prove that S generates P. Since $P' \leq \Phi(P)$, each element of S is congruent module $\Phi(P)$ to an element of $N_G(Q)$. If follows that P is generated by elements in $N_G(Q)$ and so (recall that $Q \lhd T_j$) Q is normal in G. By applying induction, we can see that G is soluble as required.

Let $R = Soc(G)$ be a p-group. Denote $C_G(R)$ by C. Then, by the minimality of $|G|$ we may assume that $\Phi(G) = 1$, hence there exists a maximal subgroup M which does not contain R. It is now clear that $C \cap M \trianglelefteq G$. Also, $M_G = 1$. This leads to $C \cap M = 1$, and thereby $C = C \cap RM = R(C \cap M) = R$. By using Lemma 2.10, we see that $O_p(G/C) = O_p(G/R) = 1$.

Clearly, $R \leq N$. Now, by using the same arguments as in the second paragraph, we see that $R \neq N$. Assume that $\pi(N) = \{p\}$. Then $N/R \leq O_p(G/R)$, and so $R = N$, a contradiction. Hence $|\pi(N)| \geq 2$.

Let d be the largest prime divisor of $|N|$. Assume that $d = p$. Since $O_p(G/R) = 1$ and $M \simeq G/R$, we have $O_p(M) = 1$. Because $|G|$ is minimal, we see that G/R is supersoluble. Hence M is also supersoluble, and so $p \notin \pi(M)$. Thus, R is a Sylow p-subgroup of G and, in particular, R is a Sylow p-subgroup of N. Therefore, if R_1 is a maximal subgroup of R, then R_1 is either normal in G or $G = R_1T$ for some supersoluble subgroup T of G. But R itself is a minimal normal subgroup of G and, evidently, $|R| \neq p$. This shows that R_1 is not normal in G. For the other case, it is impossible in view of the second paragraph. Hence $p \neq d$.

Let $x \in \pi(N) \setminus \{p\}$ and let X be a Sylow x-subgroup of N. If X_1 is a maximal subgroup of X, then X_1 is either normal in G or there exists a supersoluble subgroup T of G such that $X_1T = G$, and so $|G : T| = x^\alpha$ for some natural number α. Assume that we have the latter case. Then $R \leq T$. Since $C = R$, for every $l \in \pi(T)\backslash\{p\}$, we have $O_l(T) = 1$. Hence by Lemma 2.11, p is the largest prime divisor of $|T|$, and so p is the largest prime divisor of $|N|$. This contradiction shows that $X_1 \trianglelefteq G$. But, evidently, $X_1 \cap R = 1$, and hence $X_1 = 1$. Consequently, for all $x \in \pi(N)\backslash\{p\}$, we can see that any Sylow x-subgroup of N is always of prime order.

Suppose that $|\pi(N)| > 2$. Because N is a soluble group, we may choose a Sylow d-subgroup D of N such that for some Sylow p-subgroup H of N, the product HD is a subgroup of N. Since $\pi(N) \neq \{p, d\}$, we have $PD \neq N$, and by the choice of G, we see that the subgroup PD is supersoluble. Now, by Lemma 2.11, we have $D \trianglelefteq PD$. But $R \subseteq H$, and so $D \subseteq C = R$. This contradiction shows that $\pi(N) = \{p, d\}$, where $p = r < q = d$. Analogously, we can also obtain a contradiction for the case $N \neq G$. Hence, $N = G$.

Let Q be a Sylow q-subgroup of M. Since M is supersoluble, we have $Q \trianglelefteq M$, and so $Q \subseteq F(M)$. If $p||F(M)|$, then since any Sylow p-subgroup Y of $F(M)$ is characteristic in $F(M)$, we see that Y is normal in M. But $O_p(M) = 1$, and so $Q = F(M)$. By using the above argument again, we can easily deduce that $|Q| = q$. Hence Q is a unique minimal normal subgroup of M and, by Lemma 2.7, we have $C_M(Q) = Q$. Now, by using Lemma 2.4, we can immediately see that if P_2 is a Sylow p-subgroup of M, then $P_2 \simeq M/C_M(Q) = M/Q$. This shows that P_2 is a cyclic group.

Let $V = RP_2$. Clearly, V is a Sylow p-subgroup of G. Let P_1 be a maximal subgroup of V such that $P_2 \leq P_1$. Then $P_1 = P_1 \cap RP_2 = P_2(P_1 \cap R)$. Clearly $R \not\subseteq P_1$. If $P_1 \trianglelefteq G$, then $P_1 \cap R = 1$, and hence $P_1 \subseteq C = R$. This contradiction shows that P_1 is not normal in G, and so there is a supersoluble subgroup T of G such that $G = P_1T$. Clearly, $q \in \pi(T)$. Now, let Z_q be a Sylow q-subgroup of T. Then Z_q is a Sylow q-subgroup of G, and so $Z_q = Q^g$ for some $g \in G$. It is clear that $M = N_G(Q)$, and hence $M^g = N_G(Z_q)$. However, because the group T is supersoluble, we can deduce that $T \subseteq N_G(Z_q) = M^g$. Let $T \neq M^g$. Because $G = P_1T$, we have $M^g = M^g \cap P_1T = T(M^g \cap P_1)$, and so there exists a Sylow p-subgroup M_p of M^g and a Sylow p-subgroup T_p of T such that $M_p = T_p(M^g \cap P_1)$. But by the above argument immediately, we see that M_p is a cyclic group. Hence either $M^g \cap P_1 = M_p$ or $T_p = M_p$. For the former case, we can let P_3 be a Sylow p-subgroup of T such that $P_3 \leq M_p$. Then $P_3 \leq P_1$. However, because $T = P_3Q_1$ for some Sylow q-subgroup Q_1 of T, we have $G = P_1T = P_1P_3Q_1 = P_1Q_1$. But $|G : Q_1| = |V| > |P_1|$, a contradiction. Hence, we have now shown that $M^g \cap P_1 \neq M_p$. Thus we only have $T_p = M_p$, and so $T = M^g$.

Let $g = my$, where $m \in M$ and $y \in R$. Then $M^g = M^{my} = M^y = P_2^y(Z_q)^y$. Since P_1 is a maximal subgroup of V, P_1 is normal in V. However, because $P_2 \leq P_1$, we have $P_2^y \leq P_1$. This shows that $G = P_1T = P_1M^g = P_1P_2^y(Z_q)^y = P_1(Z_q)^y$, and therefore $|G| = |P_1|q < |V|q$. This contradiction completes our proof. ∎

Analogously, by using similar arguments, we can prove the following theorem for the class of nilpotent groups.

COROLLARY 3.8: *Let G be a group having a normal subgroup N such that G/N is nilpotent. Assume that there exists a set $\mathcal{F}$ of subgroups of G having the following property: For every maximal subgroup M of any Sylow subgroup of N the set $\mathcal{F}$ contains a supplement of M in G. Then $\mathcal{F}$ forms a G-covering subgroup system for the class of nilpotent groups.*

COROLLARY 3.9: *Let S be an arbitrary Sylow subgroup of a group G. Then for every maximal subgroup T of S, the set of every subgroup of G contains at least one supplement of T in G, and, moreover, this set forms a G-covering subgroup system for the classes of nilpotent groups and supersoluble groups.*

We now call a subgroup H of a group G Φ-free in G if G does not contain any subgroup T such that $H \subseteq \Phi(T)$.

Remark 3.10: The example of the group A_4 shows that the set of all complements of the Sylow subgroups of a group G and the set of all complements in G of the Sylow subgroups of $F(G)$ are in general not G-covering subgroup systems for the class of supersoluble groups. The example of the group S_3 shows that there exist groups G in which the set of all complements of minimal subgroups is not a G-covering subgroup system for the class of nilpotent groups. In this connection, we have the following interesting theorem.

THEOREM 3.11: *Let G be a group. Assume that there exists a set $\mathcal{F}$ of subgroups of G having the following property: For every Φ-free cyclic subgroup L of G with prime order or with order 4, either L is normal in G or $\mathcal{F}$ contains a supplement of L in G. Then $\mathcal{F}$ forms a G-covering subgroup system for the class of supersoluble groups.*

Proof: Assume that the theorem is false and let G be a counterexample with minimal order. We first let H be a subgroup of G. If x is an element in H of prime order or order 4 and T is a supersoluble subgroup of G such that $< x > T = G$, then $H = H\cap < x > T =< x > (H \cap T)$. In this case, $< x >$ has a supersoluble supplement $H \cap T$ in H. Hence, by the choice of G, this group is not supersoluble but all proper subgroups of G are supersoluble. By using Theorem 22 [7] and the main result from [3] (see, also, [8; p. 721]), we see that the group G has a normal Sylow p-subgroup P satisfying the following properties:

(1) P is the least normal subgroup of G with a supersoluble quotient group;

(2) $P/\Phi(P)$ is a chief factor of G;

(3) P has an exponent p when p is odd and has an exponent 2 or 4 when $p = 2$;

(4) $P' = \Phi(P) = P \cap \Phi(G)$, and either P is an elementary abelian subgroup or $P' = Z(P)$.

Now, let $x \in P\backslash\Phi(P)$ and $L =< x >$. We first show that L is a Φ-free subgroup of G. Assume that there exists a subgroup M of G such that $x \in \Phi(M)$. Since for every subgroup K of every nilpotent group X, we have $\Phi(K) \subseteq \Phi(X)$, and because $x \notin \Phi(P)$, we can immediately see that $M \not\subseteq P$. Also, it is clear that $M_p = P \cap M$ is a normal Sylow p-subgroup of M. Since the group G is soluble, M has a subgroup D such that $M = [M_p]D$. If M_p is an elementary p-group, then by the well known Maschke's theorem [6; Ch.3, Theorem 3.1], we have $M_p = M_1 \times \cdots \times M_t$, where each M_i is a minimal normal subgroup of M. Hence $\Phi(M) \cap M_p = 1$. But $x \in M_p \cap \Phi(M)$; this contradiction shows that $\Phi(M_p) \neq 1$. We now claim that $\Phi(M) \cap M_p = \Phi(M_p)$. Indeed, $\Phi(M_p) \subseteq \Phi(M)$. But since the Sylow p-subgroup $M_p/\Phi(M_p)$ of the factor group $M/\Phi(M_p)$ is an elementary abelian p-group, we see that $M_p/\Phi(M_p)$ is complemented in $M/\Phi(M_p)$ by the subgroup $D\Phi(M_p)/\Phi(M_p)$. This leads to $\Phi(M/\Phi(M_p)) \cap (M_p/\Phi(M_p)) = (\Phi(M) \cap M_p)/\Phi(M_p) = 1$. Consequently, $\Phi(M) \cap M_p = \Phi(M_p)$ and our claim is established. This leads to $x \in \Phi(M_p) \subseteq \Phi(P)$, which is a contradiction. Hence L is a Φ-free subgroup of G.

Now, we assume that $L \trianglelefteq G$. Then $L\Phi(P)$ is clearly a normal subgroup of G such that $\Phi(P) < L\Phi(P) \leq P$, and so $L\Phi(P) = P$. In this case, we see that the chief factor $P/\Phi(P)$ is isomorphic to the factor $L/\Phi(P) \cap L$. Since $P/\Phi(P)$ is itself an elementary abelian group, we have $\Phi(L) \subseteq \Phi(P) \cap L$. Hence $L/\Phi(P) \cap L$ is a group of prime order, and so $P/\Phi(P)$ must be a cyclic group. Since $G/P \simeq (G/\Phi(P))/(P/\Phi(P))$ is a supersoluble group, $G/\Phi(G)$ is supersoluble. This contradiction shows that L is not normal in G. However, by the choice of G, we easily see that there is a supersoluble subgroup T of G such that $G = LT$. Because G is not supersoluble, we know that $T \neq G$. Therefore, we can let H be a maximal subgroup of G containing T, that is, $T \leq H$. Now, if $P \leq H$, then

$L \leq H$, and hence $G = LT = LH = H$, again a contradiction. This shows that $P \not\leq H$. As $P \trianglelefteq G$, we have $\Phi(P) \subseteq \Phi(G)$. Consequently, we have $\Phi(P) \subseteq H$, and so $P \cap H = P \cap H_G = \Phi(P)$. This implies that the chief factor $P/\Phi(P)$ is indeed G-isomorphic to the factor $H_G P/H_G$. However, G/H_G is a primitive group and, by Lemma 2.9, PH_G/H_G is a unique minimal normal subgroup of G/H_G. Hence, it follows that $|G/H_G : H/H_G| = |PH_G/H_G| = |P/\Phi(P)| = |G : H|$. Because $LH = G$, we have $|G : H| \leq |L|$. In particular, we notice that if $|L| = 4$, then $x^2 \in \Phi(P)$, and hence $H \cap L \neq 1$. In this case, we obtain $|H \cap L| = 2$, and thereby $|G : H| = 2$. Thus, in any case, $|P/\Phi(P)| = |G : H|$ is a prime number. This contradiction completes the proof. ∎

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